Math 230: Applied Calculus II Quiz #3 Solutions

Week #4

(1) Generally, an antiderivative of $f(x) = x^n \ln(x)$ is $F(x) = \frac{1}{(n+1)^2}\left[(n+1)\ln(x) - 1\right]$. One can see this for each version, using the ntegration by parts formula $\boxed{\int \mathbf{u\,dv} = \mathbf{uv} - \int \mathbf{v\,du}}$, as follows.

$$\begin{aligned}
&f(x) = x^3 \ln(x)\\
&u = \ln(x) \rightarrow du = \frac{1}{x}dx\\
&dv = x^3 dx \rightarrow v = \frac{x^4}{4}\\
&\int x^3 \ln(x)\,dx\\
=\;& \ln(x)\cdot \quad \frac{x^4}{4} \quad - \int \frac{x^4}{4} \quad \frac{1}{x} \quad dx\\
=\;& \frac{x^4}{4}\ln(x) - \frac{1}{4}\int x^3\,dx\\
=\;& \frac{x^4}{4}\ln(x) - \frac{1}{4} \quad \frac{x^4}{4}\\
=\;& \frac{x^4}{16}\left[4\ln(x) - 1\right] + C
\end{aligned}$$

$$\begin{aligned}
&f(x) = x^4 \ln(x)\\
&u = \ln(x) \rightarrow du = \frac{1}{x}dx\\
&dv = x^4 dx \rightarrow v = \frac{x^5}{5}\\
&\int x^4 \ln(x)\,dx\\
=\;& \ln(x)\cdot \quad \frac{x^5}{5} \quad - \int \frac{x^5}{5} \quad \frac{1}{x} \quad dx\\
=\;& \frac{x^5}{5}\ln(x) - \frac{1}{5}\int x^4\,dx\\
=\;& \frac{x^5}{5}\ln(x) - \frac{1}{5} \quad \frac{x^5}{5}\\
=\;& \frac{x^5}{25}\left[5\ln(x) - 1\right] + C
\end{aligned}$$

$$\begin{aligned}
&f(x) = x^2 \ln(x)\\
&u = \ln(x) \rightarrow du = \frac{1}{x}dx\\
&dv = x^2 dx \rightarrow v = \frac{x^3}{3}\\
&\int x^2 \ln(x)\,dx\\
=\;& \ln(x)\cdot \quad \frac{x^3}{3} \quad - \int \frac{x^3}{3} \quad \frac{1}{x} \quad dx\\
=\;& \frac{x^3}{3}\ln(x) - \frac{1}{3}\int x^2\,dx\\
=\;& \frac{x^3}{3}\ln(x) - \frac{1}{3} \quad \frac{x^3}{3}\\
=\;& \frac{x^3}{9}\left[3\ln(x) - 1\right] + C
\end{aligned}$$

For this problem, as an antiderivative was requested, the additive constant $(+C)$ was not required.

(2) These quarter-ellipses satisfy the equations $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ or $y = \sqrt{b^2 - \frac{b^2}{a^2}x^2}$ and have a maximum height when x=0 of y=b and bound a region from $x = 0$ until $x = a$. For all such ellipses

$$A = \frac{\pi ab}{4},\ \overline{x} = \frac{4a}{3\pi},\ \text{and}\ \overline{y} = \frac{4b}{3\pi}.$$

To see that these formulas work for each version, recall that the formulas for the centroid's coordinates are

$$\overline{x} = \frac{1}{A}\int x f(x)\,dx \text{ and } \overline{y} = \frac{1}{A}\int \frac{1}{2}\left[f(x)\right]^2\,dx$$

For each version, this gives

$$\begin{aligned}
A &= 3\pi\\
f(x) &= \sqrt{36 - 9x^2} = 3\sqrt{4 - x^2}\\
\overline{x} &= \frac{1}{3\pi}\int_0^2 x\sqrt{36 - 9x^2}\,dx\\
&= \frac{1}{\pi}\int_0^2 x\sqrt{4 - x^2}\,dx\\
u &= 4 - x^2 \rightarrow du = -2x\,dx\\
x &= \{0, 2\} \rightarrow u = \{4, 0\}\\
\overline{\mathbf{x}} &= \frac{1}{\pi}\int_4^0 \sqrt{u} \quad -\frac{1}{2} \quad du\\
&= \frac{1}{\pi}\frac{1}{2}\int_0^4 u^{\frac{1}{2}}\,du = \frac{1}{\pi}\frac{1}{2}\frac{2}{3}u^{\frac{3}{2}} \Big|_0^4\\
&= \frac{1}{\pi}\frac{1}{3}(4)^{\frac{3}{2}} = \frac{\mathbf{8}}{\mathbf{3\pi}}.
\end{aligned}$$

$$\begin{aligned}
A &= 2\pi\\
f(x) &= \sqrt{16 - 4x^2} = 2\sqrt{4 - x^2}\\
\overline{x} &= \frac{1}{2\pi}\int_0^2 x\sqrt{16 - 4x^2}\,dx\\
&= \frac{1}{\pi}\int_0^2 x\sqrt{4 - x^2}\,dx\\
u &= 4 - x^2 \rightarrow du = -2x\,dx\\
x &= \{0, 2\} \rightarrow u = \{4, 0\}\\
\overline{\mathbf{x}} &= \frac{1}{\pi}\int_4^0 \sqrt{u} \quad -\frac{1}{2} \quad du\\
&= \frac{1}{\pi}\frac{1}{2}\int_0^4 u^{\frac{1}{2}}\,du = \frac{1}{\pi}\frac{1}{2}\frac{2}{3}u^{\frac{3}{2}} \Big|_0^4\\
&= \frac{1}{\pi}\frac{1}{3}(4)^{\frac{3}{2}} = \frac{\mathbf{8}}{\mathbf{3\pi}}.
\end{aligned}$$

$$\begin{aligned}
A &= \frac{27}{4}\pi\\
f(x) &= \sqrt{81 - 9x^2} = 3\sqrt{9 - x^2}\\
\overline{x} &= \frac{1}{\frac{27}{4}\pi}\int_0^3 x\sqrt{81 - 9x^2}\,dx\\
&= \frac{4}{9\pi}\int_0^3 x\sqrt{9 - x^2}\,dx\\
u &= 9 - x^2 \rightarrow du = -2x\,dx\\
x &= \{0, 3\} \rightarrow u = \{9, 0\}\\
\overline{\mathbf{x}} &= \frac{4}{9\pi}\int_9^0 \sqrt{u} \quad -\frac{1}{2} \quad du\\
&= \frac{4}{9\pi}\frac{1}{2}\int_0^9 u^{\frac{1}{2}}\,du = \frac{4}{9\pi}\frac{1}{2}\frac{2}{3}u^{\frac{3}{2}} \Big|_0^9\\
&= \frac{4}{9\pi}\frac{1}{3}(9)^{\frac{3}{2}} = \frac{\mathbf{4}}{\pi}.
\end{aligned}$$

$$\begin{aligned}
\overline{\mathbf{y}} &= \frac{1}{3\pi}\frac{1}{2}\int_0^2 \left[\sqrt{36 - 9x^2}\right]^2\,dx\\
&= \frac{3}{2\pi}\int_0^2 4 - x^2\,dx\\
&= \frac{3}{2\pi} \quad 4x - \frac{x^3}{3} \quad \Big|_0^2 = \frac{3}{2\pi} \quad 4(2) - \frac{(2)^3}{3}\\
&= \frac{3}{2\pi}\cdot\frac{16}{3} = \frac{\mathbf{8}}{\pi}
\end{aligned}$$

$$\begin{aligned}
\overline{\mathbf{y}} &= \frac{1}{2\pi}\frac{1}{2}\int_0^2 \left[\sqrt{16 - 4x^2}\right]^2\,dx\\
&= \frac{1}{\pi}\int_0^2 4 - x^2\,dx\\
&= \frac{1}{\pi} \quad 4x - \frac{x^3}{3} \quad \Big|_0^2 = \frac{1}{\pi} \quad 4(2) - \frac{(2)^3}{3}\\
&= \frac{1}{\pi}\cdot\frac{16}{3} = \frac{\mathbf{16}}{\mathbf{3\pi}}
\end{aligned}$$

$$\begin{aligned}
\overline{\mathbf{y}} &= \frac{1}{\frac{27}{4}\pi}\frac{1}{2}\int_0^3 \left[\sqrt{81 - 9x^2}\right]^2\,dx\\
&= \frac{2}{3\pi}\int_0^3 9 - x^2\,dx\\
&= \frac{2}{3\pi} \quad 9x - \frac{x^3}{3} \quad \Big|_0^3 = \frac{2}{3\pi} \quad 9(3) - \frac{(3)^3}{3}\\
&= \frac{2}{3\pi}\cdot\frac{54}{3} = \frac{\mathbf{12}}{\pi}
\end{aligned}$$